Hall Conductivities for Confined System in Noncommutative Plane

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Abstract

We propose an approach based on the generalized quantum mechanics to deal with the basic features of the spin Hall effect. We begin by considering two decoupled harmonic oscillators on the noncommutative plane and determine the solutions of the energy spectrum. We realize two algebras in terms of the quadratic observables and show their importance in filling the shells with fermions. Under some transformation we show that our system is submitted to an effective Lorentz force similar to that acting on one particle in an external magnetic field. From equation of motions, we end up with the charge and spin Hall conductivities as function of the noncommutative parameter θ. By switching off θ we recover standard results developed on the subject and in the limit $\theta \longrightarrow 0$ we show that our approach can reproduce the Laughin wavefunctions.

PACS numbers: 03.65.-w, 02.40.Gh, 71.70.Ej

Keywords: two harmonic oscillators, noncommutative space, spin Hall effect, Laughlin states

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1 Introduction

In 1971 D'yakonov and Perel predicted from a phenomenological model that spin-orbit coupling should lead to a new family of Hall effects [1, 2], known actually as the spin Hall effect (SHE). It is a consequence of an applied electric field to a sample that leads to a spin transport in perpendicular direction and spin accumulation at the lateral edges [3, 4]. Its most remarkable feature is a 2D semiconductor subject to a difference of the potential, which can split the spin to up and down components. The spin current produces an accumulation of spin-up electrons on one side of the excited region and spin-down electrons on the other side, with no total carrier accumulation on either side. SHE is characterized by a spin Hall conductivity resulting from the spin polarization on the boundaries of the sample. Spin-orbit effects are generally classified as extrinsic effects because they arise from scattering of impurity potentials. Another class of spin-orbit effects, known as intrinsic effects, is a consequence of the inherent band structure of a crystalline material. These effects can lead to a new state of matter called the quantum SHE, which has been predicted theoretically in 2000 by the Kane-Mele model [5] and the experimental realization came later on [6]. The innovative aspect of the quantum SHE is that it appears in the absence of magnetic field and then there is no symmetry breaking of time reversal.

On the other hand, the noncommutative geometry [7] plays an important role in physics and was used to solve many issues in different areas. For instance, interesting results were reported for the quantum Hall effect [8] due either to the charge current [9] or spin current [10–13]. To remember, the noncommutative geometry is already exits and found its application in the fractional quantum Hall effect when the lowest Landau Level (LLL) is partially filled. It happened that in LLL, the potential energy is strong enough than kinetic energy and therefore the particles are glue in the fundamental level. As a consequence of this drastic reduction of the degrees of freedom, the two space coordinates become noncommuting [14] and satisfy the commutation relations analogue to those verifying by the position and the momentum in quantum mechanics. In this case, the electron is not a point like particle anymore and can at best be localized at the scale of the magnetic length. In the present work, we serve form the mentioned mathematical tool to expose our idea and thus offer another way to study SHE for a confined system in two dimensions.

Based on the results presented in [10, 15], we quantum mechanically develop an approach to study SHE for particles living on the noncommutative plane $\mathbb{R}^2_\theta$. For this, we use the star product to define the Hamiltonian system that captures the basic features of the effect and allows for building an effective theory. This Hamiltonian is nothing but that describing two coupled harmonic oscillators where the solutions of the energy spectrum are algebraically derived as functions of the noncommutative parameter θ. Different interpretations are presented in order to give some experimental evidence of θ and show its relevance in modern physics. After realizing the algebras $su(2)$ and $su(1,1)$, we discuss the possibilities how to fill the shells with fermions. Using the Hamilton canonically equations to get the average velocity of particles, we explicitly determine the charge and spin Hall conductivities in terms of θ. The ratio between these two conductivities shows an independency of θ and the concentration of charge carriers, which is in agreement with the results obtained in [15]. Finally interesting cases are examined and in particular the limiting case $\theta \longrightarrow 0$ is analyzed to show that θ can be identified to an external magnetic field and therefore make contact with Laughin states [16].

The present paper is organized as follows. In section 2, we consider two decoupled harmonic oscillators on the noncommutative plane $\mathbb{R}^2_\theta$. This process allows us to end up with a Hamiltonian system similar to that of one particle living on the plane in the presence of an external magnetic field, known as the Landau problem. In section 3, we introduce the annihilation and creation operators to easily determine the eigenvalues and eigenstates. In section 4, we show that there are two algebras those can be used in filling the shells with fermions. We define an effective force as function of θ similar to the Lorentz force acting on one particle in section 5. By establishing the equations of motions describing our system we use the current definition to obtain the Hall conductivities in terms of θ in section 6. By fixing θ, we show that it is possible to recover interesting results dealing with SHE as well as quantum Hall effect in section 7. We conclude our work in the final section.

2 Two harmonic oscillators in noncommutative space

We consider two decoupled harmonic oscillators on the noncommutative plane $\mathbb{R}^2_\theta$ and determine the corresponding eigenvalues as well as eigenstates. In doing so, we introduce the star product and the ordinary commutation relations in quantum mechanics. Before doing this, we recall the algebraic structures of the ordinary system on the plane $\mathbb{R}^2$ and the solution of the energy spectrum.

As we claimed before our proposal can be elaborated by considering two decoupled harmonic oscillators having the same masses m and frequencies ω. These are described by the Hamiltonian

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{m\omega^2}{2}\left(x^2 + y^2\right) \tag{1}$$

which is also the Hamiltonian describing single particle living on $\mathbb{R}^2$ and subjected to a confining potential. It can be diagonalized by introducing the creation and annihilation operators

$$b_i = \frac{1}{\sqrt{2\hbar m\omega}}p_i - i\sqrt{\frac{m\omega}{2\hbar}}, \qquad b_i^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}p_i + i\sqrt{\frac{m\omega}{2\hbar}} \tag{2}$$

where $i = x, y$ and the only non-vanishing commutator are

$$\left[b_i, b_i^\dagger\right] = \mathbb{I}. \tag{3}$$

Now we can use these operators to map the Hamiltonian (1) as

$$H = \frac{\hbar\omega}{2}\left(b_x^\dagger b_x + b_y^\dagger b_y + 1\right). \tag{4}$$

Clearly from the eigenvalue equation $H|n_x, n_y\rangle = E_{n_x,n_y}|n_x, n_y\rangle$, one can easily get the eigenstates

$$|n_x, n_y\rangle = \frac{(b_x^\dagger)^{n_x}}{\sqrt{n_x!}}\frac{(b_y^\dagger)^{n_y}}{\sqrt{n_y!}}\,|0,0\rangle \tag{5}$$

as well as the eigenvalues

$$E_{n_x,n_y} = \frac{\hbar\omega}{2}\left(n_x + n_y + 1\right), \qquad n_i = 0, 1, 2, \cdots \tag{6}$$

where $|0,0\rangle$ is the fundamental state. Next, we will see how these results will be generalized for one particle living on $\mathbb{R}^2_\theta$.

Our main goal is to investigate the basic features of particles living on the noncommutative plane and in particular study SHE. To do this task, we need to settle a theoretical model that allows us to shed light on different issues. We adopt a method similar to that used in [9] where the canonical quantization of the system described by the Hamiltonian (1) is achieved by introducing the coordinate r_j and momentum p_k operators satisfying the commutation relation

$$[r_j, p_k] = i\hbar\delta_{jk}. \tag{7}$$

But to deal with our proposal, we consider a generalized quantum mechanics governed by (7) and the noncommutative coordinates

$$[x, y] = i\theta \tag{8}$$

where θ is a real free parameter and has length square of dimension. Without loss of generality, hereafter we assume that $\theta > 0$ is fulfilled. Noncommutativity can be imposed by treating the coordinates as commuting but requiring that composition of their functions is given in terms of the star product

$$\star \equiv \exp \frac{i\theta}{2} \left(\overleftarrow{\partial_x}\overrightarrow{\partial_y} - \overleftarrow{\partial_y}\overrightarrow{\partial_x} \right). \tag{9}$$

Now, we deal with the commutative coordinates x and y but replace the ordinary products with the star product (9). For example, instead of the commutator (8) one defines

$$x \star y - y \star x = i\theta. \tag{10}$$

At this level, let us derive the corresponding form of the Hamiltonian (1) in terms of the noncommutative coordinates (8). First, we quantize the present system by establishing the commutation relation (7). Second, we take into account the noncommutativity of the coordinates by defining a new Hamiltonian operator as

$$H \star \psi(\vec{r}) \equiv H^{\mathsf{nc}}\psi(\vec{r}) \tag{11}$$

where $\psi(\vec{r})$ is an arbitrary eigenfunction of H.

It is clear now that from the above materials we can easily write the noncommutative version of the Hamiltonian (1). This is

$$H^{\mathsf{nc}} = \frac{1}{2m_\theta}\left(p_x^2 + p_y^2\right) + \frac{m\omega^2}{2}\left(x^2 + y^2\right) + \frac{m\omega^2}{2\hbar}\theta\left(yp_x - xp_y\right) \tag{12}$$

where the effective mass is given by

$$m_\theta = \frac{m}{1 + \left(\frac{m\omega\theta}{2\hbar}\right)^2} \tag{13}$$

This form of mass suggest an interesting way to measure the noncommutative parameter and give hint ont its values. In doing so, one can make comparison with a relativistic particle of mass m_0 and moving with a velocity v, which acquires an effective mass

$$m_{\mathsf{eff}} = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{14}$$

Now by identifying both masses and requiring $m = m_0$, one can immediately fix θ in terms of speed of light and v as

$$\theta_\pm = \pm\frac{2\hbar}{m\omega}\left(-1 + \sqrt{1 - \frac{v^2}{c^2}}\right)^{1/2} \tag{15}$$

We have some remarks in order. Indeed, we emphasis that due to the noncommutativity between spacial coordinates we ended up with two coupled harmonic oscillators (12) where the coupling is described by the angular momenta $L_z(\theta)$ term

$$L_z(\theta) = \frac{m\omega^2\theta}{2\hbar}\left(yp_x - xp_y\right) \tag{16}$$

and then we can split H^{nc} into free part and $L_z(\theta)$

$$H^{\mathsf{nc}} = H_0^{\mathsf{nc}} + L_z(\theta) \tag{17}$$

It is obvious that (16) disappears once we set $\theta = 0$ and then we recover (1). We notice that H^{nc} is sharing some common features with the Hamiltonian describing one particle moving in plane and subjected to an external magnetic field. This statement will be clarified later on and will give another interesting feature of θ.

3 Solution of the energy spectrum

In order to get the eigenvalues and eigenstates for our system, we proceed by making use of the algebraic method to diagonalize the Hamiltonian (12). For this, we introduce a pair of the annihilation and creation operators

$$a_x = \frac{x}{l_\theta} + i\frac{l_\theta}{2\hbar}p_x, \qquad a_x^\dagger = \frac{x}{l_\theta} - i\frac{l_0}{2\hbar}p_x \tag{18}$$

$$a_y = \frac{y}{l_\theta} + i\frac{l_\theta}{2\hbar}p_y, \qquad a_y^\dagger = \frac{y}{l_\theta} - i\frac{l_0}{2\hbar}p_y \tag{19}$$

where we have defined the noncommutative length as

$$l_\theta = \sqrt[4]{\left(\frac{2\hbar}{m\omega}\right)^2 + \theta^2}. \tag{20}$$

Actually, this can be compared to the magnetic length $l_B = \sqrt{\frac{e\hbar}{cB}}$ in order to extract more information about our system, we will return to discuss this point later. One can easily show the commutation relations

$$\left[a_x, a_x^\dagger\right] = \left[a_y, a_y^\dagger\right] = \mathbb{I}. \tag{21}$$

Now let us set a new pair of the shell operators

$$a_d = \frac{1}{\sqrt{2}}(a_x - ia_y), \qquad a_d^\dagger = \frac{1}{\sqrt{2}}(a_x^\dagger + ia_y^\dagger) \tag{22}$$

$$a_g = \frac{1}{\sqrt{2}}(a_x + ia_y), \qquad a_g^\dagger = \frac{1}{\sqrt{2}}(a_x^\dagger - ia_y^\dagger) \tag{23}$$

verifying

$$\left[a_d, a_d^\dagger\right] = \left[a_g, a_g^\dagger\right] = \mathbb{I} \tag{24}$$

and all other commutators are vanishing. After some algebra, we can map the phase space coordinates in terms of the above operators as

$$x = \frac{l_0}{2\sqrt{2}}(a_d + a_d^\dagger + a_g + a_g^\dagger), \qquad p_x = \frac{\hbar}{\sqrt{2}il_0}(a_d - a_d^\dagger + a_g - a_g^\dagger) \tag{25}$$

$$y = \frac{l_0}{2\sqrt{2}i}(a_d^\dagger - a_d + a_g - a_g^\dagger), \qquad p_y = \frac{\hbar}{\sqrt{2}l_0}(a_d + a_d^\dagger - a_g - a_g^\dagger). \tag{26}$$

Combining all as well as introducing the operator numbers $N_d = a_d^\dagger a_d$ and $N_g = a_g^\dagger a_g$, we show that the Hamiltonian (12) takes the form

$$H^{\mathsf{nc}} = \frac{m\omega^2}{2}(l_\theta^2 + \theta)N_d + \frac{m\omega^2}{2}(l_\theta^2 - \theta)N_g + \frac{m\omega^2}{2}l_\theta^2. \tag{27}$$

In order to determine the solution of the energy spectrum corresponding to (27), we solve the eigenvalue equation

$$H^{\mathsf{nc}}|n_d, n_g\rangle = E_{n_d,n_g}|n_d, n_g\rangle \tag{28}$$

to end up with the eigenvalues

$$E_{n_d,n_g} = \frac{m\omega^2 l_\theta^2}{2}(n_d + n_g + 1) + \frac{m\omega^2\theta}{2}(n_d - n_g) \tag{29}$$

and eigenstates

$$|n_d, n_g\rangle = \frac{(a_d^\dagger)^{n_d}(a_g^\dagger)^{n_g}}{\sqrt{(n_d!)(n_g!)}}|0,0\rangle \tag{30}$$

where the quantum numbers n_d and n_g are non-negative integers. At this stage, we have two remarks in order. Firstly after making comparison, one can interpret the second part in (29) as a quantum correction to decoupled spectrum E_{n_x,n_y} (6). Secondly in the limit $\theta \longrightarrow 0$, (29) reduces to that of one particle in an external magnetic field B corresponding to the symmetric gauge where θ will play the role of B.

4 Algebras and filling shells

Based on the work [17], we show that the noncommutativity of our system allows us to realize two distinct dynamical symmetries in terms of the quadratic observables. Indeed, the generators of the algebra $su(2)$ can be constructed as

$$S_+ = a_d^\dagger a_g, \qquad S_- = a_g^\dagger a_d, \qquad S_z = \frac{N_d - N_g}{2} = \frac{L_z}{2\hbar} \tag{31}$$

which satisfy the $su(2)$ commutation relations

$$[S_+, S_-] = 2S_z, \qquad [S_z, S_\pm] = \pm S_\pm, \tag{32}$$

and the invariant Casimir operator is given by

$$\mathcal{C} = \left(\frac{N_d + N_g}{2}\right)\left(\frac{N_d + N_g}{2} + 1\right). \tag{33}$$

Therefore, for a fixed value $j = (n_d + n_g)/2$ of the operator $(N_d + N_g)/2 = \mathcal{H}_0^{\mathsf{nc}}/(m\omega l_\theta^2) - 1/2$, there exists a $(2j+1)$-dimensional unitary irreducible representation (UIR) of $su(2)$ in which the operator S_z has its spectral values in the range $-j \leq k \leq j$, with $k = (n_d - n_g)/2$.

Also we have another free room for a second algebra that is $su(1,1)$. The associated generators can be defined as

$$T_+ = a_d^\dagger a_g^\dagger, \qquad T_- = a_g a_d, \qquad T_0 = \frac{\mathcal{H}_0^{\mathsf{nc}}}{m\omega l_\theta^2} \tag{34}$$

showing the algebra

$$[T_+, T_-] = -2T_0, \qquad [T_0, T_\pm] = \pm T_\pm. \tag{35}$$

and the Casimir operator is given by

$$\mathcal{D} = -\left(\frac{L_z^2(\theta)}{(m\omega^2\theta)^2} - \frac{1}{4}\right). \tag{36}$$

There are two interesting situation one has to consider. First when $n_d \geq n_g$, for a fixed value $k+1/2 \geq 1/2$ of the operator $L_z(\theta)/(m\omega^2\theta)+1/2$, there exists a UIR of $su(1,1)$ in the discrete series, in which the operator T_0 has its spectral values in the infinite range $k+1/2, k+3/2, k+5/2, \cdots$. Second when $n_d \leq n_g$, for a fixed value $-k+1/2 \geq 1/2$ of the operator $-L_z(\theta)/(m\omega^2\theta)+1/2$, there also exists a UIR of $su(1,1)$ in which the spectral value of the operator T_0 runs in the infinite range $-k+1/2, -k+3/2, -k+3/2, \cdots$.

According to the above two algebras, we can offer a way how to order the quantum numbers n_d and n_g. In what follows, we consider three interesting cases where in the week and strong limits we will see the manifestation of $su(2)$ and $su(1,1)$, respectively. In the generic case, we show under circumstance it is possible to have degeneracy of filling the levels.

▶ For the week case we assume that the product fulfilled $\omega\theta \ll 1$. Let us write the eigenvalues in terms of the algebra $su(2)$, such as

$$E_{n_d,n_g} \equiv E_{j,k} = m\omega^2 l_\theta^2 j + m\omega^2\theta k + \frac{1}{2}m\omega^2 l_\theta^2 \tag{37}$$

Then in the present case it reduces as

$$E_j \approx \hbar\omega(2j+1) \tag{38}$$

which is actually θ-independent. In the present case, $su(2)$ becomes a true symmetry of the Hamiltonian, which explains the degeneracy of order $2j+1$ for the level E_j. Note that there are $(j_0+1)(2j_0+1)$ (spinless) electrons which fill the shells up to j_0.

▶ For strong case, we consider the opposite limit, i.e. $\omega\theta \gg 1$. Therefore the energy can be approximated as

$$E_{n_d} \approx m\omega^2\theta\left(n_d + \frac{1}{2}\right). \tag{39}$$

which tells us, for a given value of n_d, that there is an infinite degeneracy labeled by n_g or by $2k \leq n_d$ where the quantum number n_d corresponds to the Landau level index. The present situation can be linked with the algebra $su(1,1)$ by claiming that for a given value of $k \leq 0$, the energy eigenstates are ladder states for the discrete series representation labeled by $-k+1/2$.

▶ In the uncommensurate generic intermediate case, which means $(l_\theta^2+\theta)/(l_\theta^2-\theta) \notin \mathbb{Q}$ and no approximation is relevant, we are faced to the problem of ordering the relatively dense (but not uniformly discrete) set of eigenvalues

$$\mathcal{E}_{n_d,n_g} \equiv \frac{E_{n_d,n_g}}{m\omega^2(l_\theta^2-\theta)/2} - \frac{l_\theta^2}{l_\theta^2-\theta} = \frac{l_\theta^2+\theta}{l_\theta^2-\theta}n_d + n_g. \tag{40}$$

However, in the commensurate case, $(l_\theta^2+\theta)/(l_\theta^2-\theta) = p/q \in \mathbb{Q}$, degeneracy can be found if one requires the condition

$$\frac{p}{q} = -\frac{n_g - n_g'}{n_d - n_d'}. \tag{41}$$

and therefore in this situation we have $E_{n_d,n_g} = E_{n_d',n_g'}$.

5 Effective Lorentz force

Based on the approach developed in [15], we begin by deriving an affective force analogue to that of Lorentz acting on one particle of charge e in an external magnetic field. This will be helpful in sense that one can solve different issues and in particular those related to the Hall effect. As it will be clear later on, one can turn out te to recover interesting results developed on the subject.

To fix our ideas, let us associate our confining potential $V(\vec{r}) = \frac{m\omega^2}{2}r^2$ to an electric field E through the usual relation $\vec{E} = -\frac{1}{e}\vec{\nabla}V(\vec{r})$, with the vector position $\vec{r} = (x, y)$. This mapping can be used to write the Hamiltonian (12) as

$$H^{\mathsf{nc}} = \frac{\vec{p}^2}{2m_\theta} + \frac{e}{2\hbar}\vec{\theta}\cdot\left(\vec{E}\times\vec{p}\right) + V\left(\vec{r}\right) \tag{42}$$

where we have defined a noncommutative vector $\vec{\theta}$ as follows

$$\vec{\theta} = \begin{pmatrix} \theta_x \\ \theta_y \\ \theta_z \end{pmatrix}. \tag{43}$$

We notice that the second term in (42) is nothing but the scalar product between $\vec{\theta}$ and the angular momenta $\vec{L} = \vec{r}\times\vec{p}$. Then by recalling the spin-orbit coupling $\vec{S}\cdot\vec{L}$, one can give another way to measure the noncommutative parameter in terms of the spin under the request $\theta_x \equiv S_x$, $\theta_y \equiv S_y$ and $\theta_z \equiv S_z$. This can be generalized to describe interesting features of the spin-orbit coupling and therefore make contact with the Rashba and Dresselhaus couplings [18], which are the cornerstones of different spin Hall effects.

Now let us examine the dynamical behavior of our system and write the corresponding equations of motions. Indeed, from (42) we show that the Hamiltonian mechanics for canonically conjugated variables $\vec{p}$ and $\vec{r}$ is governed by the equations

$$\dot{\vec{r}} = \frac{\vec{p}}{m_\theta} + \frac{e}{2\hbar}\left[\vec{\theta}\times\vec{E}\right] \tag{44}$$

$$\dot{\vec{p}} = -\frac{\partial V(\vec{r})}{\partial\vec{r}} - \frac{e}{2\hbar}\frac{\partial}{\partial\vec{r}}\left(\left[\vec{\theta}\times\vec{E}\right]\cdot\vec{p}\right) \tag{45}$$

leading to the relations

$$\vec{p} = m_\theta\dot{\vec{r}} - \frac{em_\theta}{2\hbar}\left[\vec{\theta}\times\vec{E}\right] \tag{46}$$

$$\dot{\vec{p}} = m_\theta\ddot{\vec{r}} - \frac{em_\theta}{2\hbar}\left(\dot{\vec{r}}\frac{\partial}{\partial\vec{r}}\right)\left[\vec{\theta}\times\vec{E}\right]. \tag{47}$$

After substituting these into (44) and (45), we end up with the following form of the second Newtons law for charge carriers

$$m_\theta\ddot{\vec{r}} = -\frac{\partial V(\vec{r})}{\partial\vec{r}} + \frac{em_\theta}{2\hbar}\left\{\left(\dot{\vec{r}}\frac{\partial}{\partial\vec{r}}\right)\left[\vec{\theta}\times\vec{E}\right] - \frac{\partial}{\partial\vec{r}}\left(\left[\vec{\theta}\times\vec{E}\right]\cdot\dot{\vec{r}}\right)\right\} + \frac{e^2m_\theta}{4\hbar}\frac{\partial}{\partial\vec{r}}\left[\vec{\theta}\times\vec{E}\right]^2. \tag{48}$$

To simplify our task and proceed further, let us introduce a convenient approximation. Indeed, we neglect the term proportional to $\left[\vec{\theta}\times\vec{E}\right]^2$ and then write the last equation as

$$m_\theta\ddot{\vec{r}} \simeq -\frac{\partial V(\vec{r})}{\partial\vec{r}} + \vec{F}(\vec{\theta}, \vec{r}, \dot{\vec{r}}) \tag{49}$$

where we have fixed $\vec{F}$ as

$$\vec{F}(\vec{\theta},\vec{r},\dot{\vec{r}}) = -\frac{em_\theta}{2\hbar}\dot{\vec{r}}\times\left(\frac{\partial}{\partial\vec{r}}\times\left[\vec{\theta}\times\vec{E}\right]\right) \tag{50}$$

Now if we look at the shape of the Lorentz force, one can immediately conclude that $\vec{F}$ is its analogue and can be mapped into

$$\vec{F}(\vec{\theta},\vec{r},\dot{\vec{r}}) = \frac{e}{c}\left(\dot{\vec{r}}\times\vec{B}(\vec{\theta})\right) \tag{51}$$

where the effective magnetic field can be defined in terms of the vector potential $\vec{A}(\vec{\theta})$ via standard relation $\vec{B}(\vec{\theta}) = \vec{\nabla}\times\vec{A}(\vec{\theta})$, such as

$$\vec{A}(\vec{\theta}) = \frac{cm_\theta}{2\hbar}\left[\vec{\theta}\times\vec{E}\right]. \tag{52}$$

Keeping in mind the Hamiltonian structures for one particle in magnetic field, known as Landau problem, we can then use the above tool to write (42) in a compact form. Indeed, after neglecting the square term in θ we find

$$H^{\mathsf{nc}} \simeq \frac{1}{2m_{\theta_z}}\left(\vec{p}+\frac{e}{c}\vec{A}(\vec{\theta})\right)^2 + V(\vec{r}). \tag{53}$$

It is clearly seen that this Hamiltonian can be interpreted as one describing a particle living on the plane in the presence of the effective magnetic field $\vec{B}(\vec{\theta})$ and scalar potential $V(\vec{r})$. Based on our knowledge, it turns out that (53) can be used as model to build an effective theory for the fractional quantum Hall effect [8] and related matters.

Now let us return to discus the product $\vec{\theta}\times\vec{E}$ and see what we can extract as information regarding our system. To fix our ideas, we choose hereafter the noncommutative components as $\theta_x=\theta_y=0$ and $\theta_z=\theta$. From the confining potential we can easily obtain the electric field

$$\vec{E} = -\frac{m\omega^2}{e}(x,y) \tag{54}$$

and then (52) gives

$$\vec{A}(\vec{\theta}) = \frac{2\hbar c\theta/e}{\left(\frac{2\hbar}{m\omega}\right)^2+\theta^2}(y,-x). \tag{55}$$

It can be compared to the symmetric gauge

$$\vec{A}(x,y) = \frac{B}{2}(y,-x). \tag{56}$$

to find a second order equation for θ

$$\theta^2 - 4l_B^2\theta + \left(\frac{2\hbar}{m\omega}\right)^2 = 0 \tag{57}$$

where $l_B=\sqrt{\frac{\hbar c}{eB}}$ is the magnetic length, which define the area occupied by the Hall droplet [8]. This can be solved to obtain

$$\theta_\pm = 2l_B^2 \pm 2\sqrt{l_B^4 - \left(\frac{\hbar}{m\omega}\right)^2} \tag{58}$$

where the condition $l_B \geq \sqrt{\frac{\hbar}{m\omega}}$ must be fulfilled. This shows another alternative way to measure and give some interpretation for θ.

6 Hall conductivities

Actually, it known that Hall effect remains among the fascinating areas appeared in condensed matter physics. It comes out that one can ask about the possibility to describe such effect in terms of language. To answer this inquiry, we will show how one can use our results to determine explicitly the corresponding charge and spin the conductivities.

In the spirit of the Drude model we shall now add to equation (49) the drag force $-m_{\theta_z}\dot{\vec{r}}/\tau$, with τ is the relaxation time. Then we can write

$$m_\theta \ddot{\vec{r}} + \frac{m_\theta}{\tau}\dot{\vec{r}} + \frac{\partial V(\vec{r})}{\partial \vec{r}} \approx \vec{F}(\vec{\theta}, \vec{r}, \dot{\vec{r}}). \tag{59}$$

In this situation $\vec{F}(\vec{\theta}, \vec{r}, \dot{\vec{r}})$ can be seen as a perturbation to the system described by the second order differential equation. Taking this into consideration, we split the solution of (59) into two parts

$$\dot{\vec{r}} = \dot{\vec{r}}_0 + \dot{\vec{r}}_1 \tag{60}$$

where the average of the quantities $\dot{\vec{r}}_0$ and $\dot{\vec{r}}_1$ are given by

$$\langle \dot{\vec{r}}_0 \rangle = \frac{\tau e}{m_\theta}\vec{E} \tag{61}$$

$$\langle \dot{\vec{r}}_1 \rangle = \frac{\tau e}{m_\theta}\langle \vec{F}(\vec{\theta}, \vec{r}_0, \dot{\vec{r}}_0)\rangle. \tag{62}$$

From (51) we obtain the relation

$$\langle \vec{F}(\vec{\theta}, \vec{r}_0, \dot{\vec{r}}_0)\rangle = \frac{e}{c}\langle \dot{\vec{r}}_0 \rangle \times \langle \vec{B}(\vec{\theta}, \vec{r}_0)\rangle. \tag{63}$$

Now using (52) together with (61) and (62) to show

$$\langle \dot{\vec{r}}_1 \rangle = \frac{e^2\tau^2}{2m_{\theta_z}\hbar}\vec{E} \times \left(\frac{\partial}{\partial \vec{r}} \times \left[\vec{\theta} \times \vec{E}\right]\right) \tag{64}$$

where the right hand side of (64) contains the volume average of $\frac{\partial \vec{E}}{\partial \vec{r}}$. This can be evaluated by considering the scalar potential $V(\vec{r}) = \frac{m\omega^2}{2}r^2$ to write $\frac{\partial \vec{E}}{\partial \vec{r}} = -\frac{m\omega^2}{e}$ and then it follows

$$\langle \dot{\vec{r}}_1 \rangle = \frac{em\tau^2\omega^2}{2\hbar m_\theta}\left[\vec{\theta} \times \vec{E}\right]. \tag{65}$$

Having established all needed necessary material, we focus now on the derivation of the Hall conductivities. For this, we begin by defining the vector of spin polarization associated to the non-commutative plane $\mathbb{R}^2_\theta$

$$\vec{\xi}(\vec{\theta}) = \langle \vec{\theta} \rangle \tag{66}$$

and absolute value ξ_θ lies between 0 and 1, such as

$$\xi_\theta = \frac{n^+ - n^-}{n^+ + n^-} \tag{67}$$

where we denote by $n^\pm$ the concentrations of charge carriers with spins parallel and antiparallel to $\vec{\xi}(\vec{\theta})$, respectively, in $\mathbb{R}^2_\theta$. It is convenient to introduce the total concentration $n = n^+ + n^-$ of charges

carrying the electric current in $\mathbb{R}^2_\theta$. The density matrix of the charge carriers in the spin space can be written as

$$N(\vec{\theta}) = \frac{1}{2}n\left(1 + \vec{\xi}(\vec{\theta})\cdot\vec{\theta}\right). \tag{68}$$

We choose the spin polarization vector $\vec{\xi}(\vec{\theta})$ along the z-direction, i.e. $\vec{\xi}(\vec{\theta}) = \xi_\theta \vec{e}_z$, and adopting the definition of the electric current [15]

$$\vec{j} = e\langle N(\vec{\theta})\dot{\vec{r}}\rangle = e\langle N(\vec{\theta})(\dot{\vec{r}}_0 + \dot{\vec{r}}_1)\rangle. \tag{69}$$

It can be evaluated by using (61) and (65) to write

$$\vec{j} = \sigma_c(\theta)\vec{E} + \sigma_s(\theta)\left(\vec{\xi}(\vec{\theta})\times\vec{\theta}\right) \tag{70}$$

and therefore the deformed charge and spin Hall conductivities read as

$$\sigma_c(\theta) = \frac{ne^2\tau}{m_\theta} \tag{71}$$

$$\sigma_s(\theta) = \frac{ne\tau^2 m\omega^2}{4m_\theta\hbar} \tag{72}$$

which are clearly noncommutative parameter θ-dependent. This will offer different discussions and interpretation in the forthcoming analysis.

Having obtained our conductivities, let us make different discussions and offer some interpretation. These will be the subject of the following points:

► By switching off θ in (71) and (72), we can easily obtain

$$\sigma_c = \frac{ne^2\tau}{m} \tag{73}$$

$$\sigma_s = \frac{ne\tau^2\omega^2}{4\hbar} \tag{74}$$

which can be identified to that obtained in [15] where ω^2 has to be taken proportional to the system area. This first connection tells us that are general so that one can extract more information about the present system by playing with the values taken by θ. More detail about this point will give in the next section.

► By looking at the ratio

$$\frac{\sigma_s(\theta)}{\sigma_c(\theta)} = \frac{\sigma_s}{\sigma_c} = \frac{m\omega^2\tau}{4e\hbar} \tag{75}$$

one realizes immediately that it is independent of the concentration of charge carriers. This is in agreement with the result obtained in different microscopic models of SHE [19, 20]. More discussion about such point and its relation with temperature can be found in [15].

► To make comparison with already published work, we write (71) and (72) in terms of the standard quantities σ_c and σ_s as

$$\sigma_c(\theta) = \sigma_c + \frac{ne^2\tau m\omega^2}{4\hbar^2}\theta^2 \tag{76}$$

$$\sigma_s(\theta) = \sigma_s + \frac{ne\tau^2 m^2\omega^4}{16\hbar^3}\theta^2 \tag{77}$$

which are parabolic functions in terms of the noncommutative parameter. It is clear that from (76) and (77), the conductivities can be controlled either by adjusting the concentration n of charge carriers or parameter θ. This latter can be regarded as an external source in similar way to the magnetic field in the systems exhibiting the quantum Hall effect [8]. Furthermore, the second terms in the above equations can be interpreted as quantum corrections to the standard results, which consists another way to look at our findings.

► The Hall conductivities obtained using different approach and techniques [11–13] are linear function of θ rather than quadratic as we have in (76) and (77). This difference is due to the manifestation of the confining potential taken into account in the noncommutative plane.

► Our spin conductivity can be identified to quantized ones obtained by using other approaches. With that one can make a quantization of the noncommutative parameter and fix some of its experimental values. Indeed on the light of the results reported in [15], we can use the experimental data on aluminum to give a table showing some of the θ values.

7 Interesting cases

Having obtained the general forms of the Hall conductivities, let us show how to recover some significant results derived in different microscopic models. Indeed, In studying SHE Chudnovsky [15] considered the following Hamiltonian in real space

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + V(\vec{r}) \tag{78}$$

where the scalar potential is defined as

$$V(\vec{r}) = \frac{4\pi\hbar^2 e^3 Z n_0}{3m^2c^2} r^2 \tag{79}$$

with -Ze and n_0 are the charge and the concentration of ions respectively. This model was used to obtain similar relations to (73) and (74). Now it clear that, these results can be derived from our approach by taking $\theta = 0$ and fixing the frequency as

$$\omega = \sqrt{\frac{8\pi\hbar^2 e^3 Z n_0}{3m^3c^2}}. \tag{80}$$

On the other hand, we can keep the noncommutative parameter and require an identification between our Hall conductivities and those obtained by Chudnovsky. This will offer another way to fix θ in terms of different parameters appearing in the potential (79) and also allow for establishing a link with the experimental data.

Another interesting theory that can be derived from our results is the Landau problem, which is describing one particle living on the plane in the presence of an uniform magnetic field B. More precisely, in the limit $\theta \longrightarrow 0$ we show that (12) has a strong overlapping with such problem and therefore it can be adopted to reproduce its basic features. Then in such limit we can approximate the effective mass (13) by writing $m_\theta \simeq m$ and now the Hamiltonian (12) reduces

$$H^{\mathsf{nc}}_{\theta\to 0} = \frac{1}{2m}\left[\left(p_x^2 + p_y^2\right) + m^2\omega^2\left(x^2 + y^2\right) + \frac{m^2\omega^2}{\hbar}\theta\left(yp_x - xp_y\right)\right] \tag{81}$$

which means that we keep only the first order term in θ and drop the second one. One can see that (81) is actually sharing some common features with the Landau problem on the plane. To clarify this statement, let us chose the symmetric gauge $A = \frac{B}{2}(-y, x)$ to write the Hamiltonian for one charged particle living on plane and in magnetic field as

$$H_{\text{landau}} = \frac{1}{2m}\left[\left(p_x^2 + p_y^2\right) + \left(\frac{eB}{2c}\right)^2 \left(x^2 + y^2\right) + \frac{eB}{c}\left(yp_x - xp_y\right)\right] \tag{82}$$

Now it clear that H_{landau} and $H^{\mathsf{nc}}_{\theta\to 0}$ are similar and then one can go from one to another by using the following mapping

$$\theta^{\text{landau}} = \frac{4\hbar c}{eB} = 4l_B^2, \qquad \omega = \frac{\omega_c}{2} \tag{83}$$

where l_B is the magnetic length and $\omega_c = \frac{eB}{mc}$ is the cyclotron frequency. Therefore θ plays actually the role of the magnetic field B, which is not surprising because it known that at LLL the position coordinates become noncommuting. This analogy will allow us to bring different results regarding B straightforward to θ. Indeed, since the filling factor of the Landau levels reads as the ratio between the density of charge carriers ρ and B

$$\nu_B = \frac{\rho ec}{B} = \frac{l_B^2 \rho}{2\pi}. \tag{84}$$

Then in similar way we write

$$\nu_\theta = \frac{\theta\rho}{8\pi}. \tag{85}$$

It gives another way to measure θ in terms of the observed quantized plateaus [8] and one illustration will be done soon.

Since (82) is the cornerstone of the quantum Hall effect [8], then $H^{\mathsf{nc}}_{\theta\to 0}$ will allow us to build a theory similar to that of the Landau problem. Let us consider N particles described by

$$\mathcal{H} = \sum_{i=1}^{N} \left(H^{\mathsf{nc}}_{\theta\to 0}\right)_i \tag{86}$$

where $\left(H^{\mathsf{nc}}_{\theta\to 0}\right)_i$ is for one particle given in (82). Starting from the above Hamiltonian, we can construct the Laughlin wavefunction [16] for the filling factor $\nu = 1$ as Slater determinant and then we generalize to write

$$\Phi^l_{\mathsf{L}}(z, \bar{z}, \theta) = \prod_{i<j}(z_i - z_j)^{2l+1} \exp\left(-\frac{1}{\theta}\sum_i |z_i|^2\right) \tag{87}$$

where $z = x + iy$ complex variable, $\nu = \frac{1}{2l+1}$ and l is integer value. Compared to the Laughlin states, we can interpret (87) as describing Hall liquid where each droplet is occupying an area of surface $\frac{\pi}{2}\theta$. Now according to (85), we can quantize θ

$$\theta'_l = \frac{\theta_l}{8\pi/\rho} = \frac{1}{2l+1} \tag{88}$$

We can go further and talk about other issues related the Laughlin wavefunctions (87) like for instance fractional charges.

8 Conclusion

A system of two harmonic oscillators with the same masses and frequencies were considered on the noncommutative plane. This theoretical model offered for us a mathematical tool to develop an approach in studying the basic features of the present system and in particular the Hall effect. More precisely, by considering plane coordinates are noncommuting, we have obtained an effective Hamiltonian involving coupling term that was interpreted as spin-orbit coupling where the noncommutative parameter θ is identified to the spin. Moreover, we have obtained a noncommutative mass m_θ, which was linked to mass of a relativistic particle moving with velocity v. This connection allowed for another possibility to give some measurement for θ and thus an experimental proof. Using the algebraic method, we have determined explicitly the solution of the energy spectrum in terms of θ. By requiring that $\theta = 0$, we have seen that the standard results can be recovered easily.

The symmetry was also taken part of our investigation. Indeed, in terms of the quadratic observables, we have realized two algebras: $su(2)$ and $su(1,1)$. For each algebra, we have discussed the corresponding unitary irreducible representations and their relations to our eigenvalues. Later on, we have analyzed three interesting cases in filling the shells with fermions: week, strong and generic coupling. The first case showed a manifestation of $su(2)$ where the level degeneracies were fixed and the filling of shells was done. While in the second case, we have found an infinite degeneracy and our eigenstates are ladder states for the discrete series representation labeled by $-k + 1/2$.

Subsequently, we have shown that it is possible to derive an effective Lorentz force as function of θ similar to that acting on one particle living on plane in the presence of magnetic field B. Therefore, the gauge field θ-dependent was established and its relation to that corresponding to B was discussed. Indeed, after identification with symmetric gauge, we have obtained a second order equation for θ, which was solved to express θ in terms of the magnetic length l_B. This allowed for another way to describe the noncommutative parameter and then linked with experimental data.

By adding a drag term to the obtained second order differential equation for the position $\vec{r}$, we were able to talk about the Hall conductivities exhibited by our system. Indeed, such equation allowed us to get the corresponding velocity that was used to calculate the electric current according to definition adopted in [15]. Doing this processing to finally find the charge and spin Hall conductivities as quadratic functions of θ. More precisely, these two conductivities showed extra terms in addition to the standard one, which were regarded as quantum corrections.

Furthermore, we have analyzed interesting limiting cases, which were lead to recover already published works. Indeed, by fixing our frequency as function of different physical parameters and switching off θ, we have recovered the Chudnovsky results. By considering the limit $\theta \longrightarrow 0$, our noncommutative Hamiltonian was reduced to another one, which has some common features with the Landau problem. This allowed us to make a mapping between both and therefore establish a bridge to Laughlin states where θ was quantized in terms of filling factor.

Acknowledgments

The generous support provided by the Saudi Center for Theoretical Physics (SCTP) is highly appreciated by all authors. AJ acknowledges partial support by King Faisal University.

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